Hawking Radiation in de Sitter Space: Calculation of the Reflection Coefficient for Quantum Particles

V. Red'kov, E. Ovsiyuk, and G.G. Krylov*

Institute of Physics, NAS of Belarus

Mozyr State Pedagogical University

Belarussian State University

Though the problem of Hawking radiation in de Sitter space-time, in particular details of penetration of a quantum mechanical particle through the de Sitter horizon, has been examined intensively there is still some vagueness in this subject. The present paper aims to clarify the situation. A known algorithm for calculation of the reflection coefficient $R_{\epsilon j}$ on the background of the de Sitter space-time model is analyzed. It is shown that the determination of $R_{\epsilon j}$ requires an additional constrain on quantum numbers $\epsilon R/\hbar c >> j$, where R is a curvature radius. When taking into account this condition, the value of $R_{\epsilon j}$ turns out to be precisely zero.

It is shown that the basic instructive definition for the calculation of the reflection coefficient in de Sitter model is grounded exclusively on the use of zero order approximation in the expansion of a particle wave function in a series on small parameter $1/R^2$, and it demonstrated that this recipe cannot be extended on accounting for contributions of higher order terms. So the result $R_{\epsilon j} = 0$ which has been obtained from examining zero-order term persists and cannot be improved.

It is claimed that the calculation of the reflection coefficient $R_{\epsilon j}$ is not required at all because there is no barrier in the effective potential curve on the background of the de Sitter space-time, the later correlate with the fact that the problem in de Sitter space reduces to a second order differential equation with only three singular points. However all known quantum mechanical problems with potentials containing one barrier reduce to a second order differential equation with four singular points, the equation of Heun class.

I. INTRODUCTION

The problem of Hawking radiation [1], and in particular the radiation in de Sitter space-time [2] and details of penetration of quantum mechanical particles through the de Sitter horizon were examined in the literature [3–10]. Till now remains some vagueness in this point, and the present

*Electronic address: redkov@dragon.bas-net.by,, e.ovsiyuk@mail.ru, krylov@bsu.by

paper aims to clarify the situation.

In the paper, exact wave solutions for a particle with spin 0 in the static coordinates of the de Sitter space-time model are examined in detail, and the procedure for calculating of the reflection coefficient $R_{\epsilon j}$ is analyzed. First, for scalar particle, two pairs of linearly independent solutions are specified explicitly: running and standing waves. A known algorithm for calculation of the reflection coefficient $R_{\epsilon j}$ on the background of the de Sitter space-time model is analyzed. It is shown that the determination of $R_{\epsilon j}$ requires an additional constrain on quantum numbers $\epsilon R/\hbar c \gg j$, where R is a curvature radius. When taking into account for this condition, the value of $R_{\epsilon j}$ turns out to be precisely zero.

It is claimed that the calculation of the reflection coefficient $R_{\epsilon j}$ is not required at all because there is no barrier in the effective potential curve on the background of the de Sitter space-time.

The same conclusion holds for arbitrary particles with higher spins, it was demonstrated explicitly with the help of the exact solutions for electromagnetic and Dirac fields in [11].

The structure of the paper is as follows. In Section 2 we state the problem; some more details concerning the approach used could be found in [11].

In Section 3 we demonstrate that the basic instructive definition for the calculation of the reflection coefficient in de Sitter model is grounded exclusively on the use of zero order approximation $\Phi^{(0)}(r)$ in the expansion of a particle wave function in a series of the form

$$\Phi(r) = \Phi^{(0)}(r) + \left(\frac{1}{R^2}\right)\Phi^{(1)}(r) + \left(\frac{1}{R^2}\right)^2 \Phi^{(2)}(r) + ... \tag{1}$$

What is even more important and we will demonstrate it explicitly, this recipe cannot be extended on accounting for contributions of higher order terms. So the result $R_{\epsilon j} = 0$ which will be obtained below from examining zero-order term $\Phi^{(0)}(r)$ persists and cannot be improved.

II. REFLECTION COEFFICIENT

Wave equation for a spin 0 particle (M is used instead of $McR/\hbar$, R is the curvature radius) reads

$$\left(\frac{1}{\sqrt{-g}}\partial_\alpha \sqrt{-g} g^{\alpha\beta}\ \partial_\beta + 2 + M^2 \right)\ \Psi(x) = 0\ , \tag{2}$$

and is considered in static coordinates

$$dS^2 = \Phi dt^2 - \frac{dr^2}{\Phi} - r^2(d\theta^2 + \sin^2\theta d\phi^2)\ , \qquad 0 \le r < 1\ , \qquad \Phi = 1 - r^2\ . \tag{3}$$

For spherical solutions $\Psi(x) = e^{-i\epsilon t} f(r) Y_{jm}(\theta,\phi)$, $\epsilon = ER/\hbar c$, the differential equation for $f(r)$ is

$$\frac{d^2 f}{dr^2} + \left(\frac{2}{r} + \frac{\Phi'}{\Phi} \right) \frac{df}{dr} + \left(\frac{\epsilon^2}{\Phi^2} - \frac{M^2+2}{\Phi} - \frac{j(j+1)}{\Phi r^2} \right) f = 0 \; . \tag{4}$$

All solutions are constructed in terms of hypergeometric functions (let $r^2 = z$):

regular at $r = 0$ standing waves are given as

$$\begin{gathered} f(z) = z^{j/2} \, (1-z)^{-i\epsilon/2} \, F(a,b,c;z) \; , \qquad \kappa = j/2 \; , \;\; \sigma = -i\epsilon/2 \; , \;\; c = j+3/2 \; , \\ a = \frac{3/2 + j + i\sqrt{M^2-1/4} - i\epsilon}{2} \; , \qquad b = \frac{3/2 + j - i\sqrt{M^2-1/4} - i\epsilon}{2} \; ; \end{gathered} \tag{5}$$

singular at $r = 0$ standing waves are

$$\begin{gathered} g(z) = z^{-(j+1)/2} \, (1-z)^{-i\epsilon/2} \, F(\alpha, \; \beta, \; \gamma; \; z) \; , \qquad \kappa = -(j+1)/2 \; , \; \sigma = -i\epsilon/2 \; , \\ c = -j + 1/2 \; , \; \alpha = \frac{1/2 - j + i\sqrt{M^2-1/4} - i\epsilon}{2} \; , \qquad \beta = \frac{1/2 - j - i\sqrt{M^2-1/4} - i\epsilon}{2} \; . \end{gathered} \tag{6}$$

With the use of the Kummer's relations, one can expand the standing waves into linear combinations of the running waves

$$f(z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \, U^{out}_{run}(z) \; + \; \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \, U^{in}_{run}(z) \; , \tag{7}$$

$$\begin{gathered} U^{out}_{run}(z) = z^{j/2} \, (1-z)^{-i\epsilon/2} \, F(a,b,a+b-c+1;1-z) \; , \\ U^{in}_{run}(z) = z^{j/2} \, (1-z)^{+i\epsilon/2} \, F(c-a,c-b,c-a-b+1;1-z) \; , \end{gathered} \tag{8}$$

$$a^* = (c-a) \; , \; b^* = (c-b) \; , \qquad (a+b-c)^* = -(a+b-c) \; ,$$

$$[\, U^{out}_{run}(z) \,]^* = U^{in}_{run}(z) \; , \tag{9}$$

$$f(z) = 2 \; \mathrm{Re} \, \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \, U_{out}(z) = 2 \;\; \mathrm{Re} \;\; \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \, U_{in}(z) \;\; . \tag{10}$$

Similarly for $g(z)$

$$g(z) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \, U^{out}_{run}(z) \; + \; \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \, U^{in}_{run}(z) \; , \tag{11}$$

$$U^{out}_{run}(z) = z^{j/2} \, (1-z)^{-i\epsilon/2} F(\alpha+1-\gamma, \beta+1-\gamma, \alpha+\beta+1-\gamma; 1-z) \; ,$$

$$U^{in}_{run}(z) = z^{j/2} \, (1-z)^{+i\epsilon/2} F(1-\alpha, 1-\beta, \gamma+1-\alpha-\beta; 1-z) \; , \tag{12}$$

$$g(z) = 2 \text{ Re } \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \; U^{out}_{run}(z) = 2 \text{ Re } \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \; U^{in}_{run}(z) \; . \tag{13}$$

Asymptotic behavior of the running waves is given by the relations

$$U^{out.}_{run}(r \sim 0) \sim \frac{1}{r^{j+1}} \; , \quad U^{out}_{run}(r \sim 1) \sim (1-r^2)^{-i\epsilon/2} \; ,$$
$$U^{in}_{run}(r \sim 0) \sim \frac{1}{r^{j+1}} \; , \qquad U^{in}_{run}(r \sim 1) \sim (1-r^2)^{+i\epsilon/2} \; , \tag{14}$$

or in new radial variable $r^* \in [0, \; \infty)$:

$$r^* = \frac{R}{2} \; \ln \frac{1+r}{1-r} \quad , \quad r = \frac{\exp(2r^*/R)-1}{\exp(2r^*/R)+1} \quad ,$$
$$U^{out}_{run}(r^* \sim \infty) \sim \left(2^{-iER/\hbar c}\right) \; \exp(+iEr^*/\hbar c) \; ,$$
$$U^{in}_{run}(r^* \sim \infty) \sim \left(2^{+iER/\hbar c}\right) \; \exp(-iEr^*/\hbar c) \; , \qquad \epsilon = ER/\hbar c) \; . \tag{15}$$

For the standing waves we have

$$f(r \sim 0) \sim r^j \; , \qquad g(r \sim 0) \sim \frac{1}{r^{j+1}} \; ,$$
$$f(r \sim 1) \sim 2 \text{ Re } \left[\; \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \; 2^{+iER/\hbar c} \exp(-iEr^*/\hbar c) \; \right] ,$$
$$g(r \sim 1) \sim 2 \text{ Re } \left[\; \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \; 2^{+iER/\hbar c} \exp(-iEr^*/\hbar c) \; \right] . \tag{16}$$

On can perform the transition to the limit of the flat space-time in accordance with the following rules:

$$a = \frac{p+1-i\epsilon R + i\sqrt{R^2M^2-1/4}}{2} \; , \qquad b = \frac{p+1-i\epsilon R - i\sqrt{R^2M^2-1/4}}{2} \; , \; p = j+1/2 \;\; ; \tag{17}$$

$$\lim_{R\to\infty} (R^2 z) = R^2, \;\; F(a,b,c;z) = 1 + \frac{ab}{c}\frac{z}{1!} \; + \; \frac{a(a+1)b(b+1)}{c(c+1)} \; \frac{z^2}{2!} \; + \; \ldots \; , \tag{18}$$

and further $(R \to \infty)$

$$\frac{a+n}{R} = \frac{1}{2}\left(\frac{p+1}{R} - i\epsilon + i\sqrt{M^2 - \frac{1}{4R^2}} \right) + \frac{n}{R} \approx \frac{-i\epsilon - iM}{2} \; ,$$
$$\frac{b+n}{R} = \frac{1}{2}\left(\frac{p+1}{R} - i\epsilon - i\sqrt{M^2 - \frac{1}{4R^2}} \right) + \frac{n}{R} \approx \frac{-i\epsilon + iM}{2} \; . \tag{19}$$

Let $\epsilon^2 - M^2 \equiv k^2$, then we arrive at

$$\lim_{R\to\infty} F(a,b,c;z) = \Gamma(1+p) \; \sum_0^\infty \frac{(-k^2R^2/4)^n}{n!\Gamma(1+n+p)} \; ,$$
$$\lim_{R\to\infty} F(b-c+1, a-c+1, -c+2; z) = \Gamma(1-p) \; \sum_0^\infty \frac{(-k^2R^2/4)^n}{n!\Gamma(1+n-p)} \; . \tag{20}$$

Allowing for the know expansion for Bessel functions

$$J_p(x) = (\frac{x}{2})^p \sum_0^\infty \frac{(ix/2)^{2n}}{n!\Gamma(1+n+p)} ,$$

we get (wave amplitude A will be determined below)

$$\lim_{R\to\infty} A\, U^{out}_{run}(z) = \lim_{R\to\infty} A\, \frac{1}{\sqrt{r}},$$

$$\times \left[\frac{\Gamma(-i\epsilon R+1)\Gamma(-p)\,\Gamma(1+p)(2/k)^p R^{-p+1/2}}{\Gamma[\frac{1}{2}(+i\sqrt{R^2M^2-1/4}-i\epsilon R+p+1)]\Gamma[\frac{1}{2}(-i\sqrt{R^2M^2-1/4}-i\epsilon R+p+1)]}\ J_p(kr) \right.$$
$$\left. + \frac{\Gamma(-i\epsilon R+1)\Gamma(+p)\Gamma(1-p)(2/k)^{-p}R^{+p+1/2}}{\Gamma[\frac{1}{2}(+i\sqrt{R^2M^2-1/4}-i\epsilon R-p+1)]\Gamma[\frac{1}{2}(-i\sqrt{R^2M^2-1/4}-i\epsilon R-p+1)]}\ J_{-p}(kr) \right] . \tag{21}$$

Performing limiting procedure (see the detail in [11]) we derive the relation

$$\lim_{R\to\infty} A\, U_{out}(z) \to \frac{1}{i^{j+1}} \sqrt{\frac{2}{kr}}\, H^{(1)}_{j+1/2}(kr) , \tag{22}$$

where

$$H^{(1)}_{j+1/2}(x) = \frac{ip}{\sin(\pi p)} \left[e^{ip\pi}\, J_p(x) \;-\; J_{-p}(x) \right]$$

stands for Hankel spherical functions.

In connection with the limiting procedure, let us pose a question: when the relation (22) gives us with a good approximation provided the curvature radius R is finite.

This point is important, because when calculating the reflection coefficient in the de Sitter space just this approximation (22) was used [3–6].

To clarify this point, let us compare the radial equation in Minkowski model

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \epsilon^2 - M^2 - \frac{j(j+1)}{r^2} \right] f^0_{\epsilon j} = 0 , \tag{23}$$

and the appropriate equation in de Sitter model

$$\left[\frac{d^2}{dr^2} + \frac{2(1-2r^2/R^2)}{r(1-r^2/R^2)}\frac{d}{dr} + \frac{\epsilon^2}{(1-r^2/R^2)^2} - \frac{M^2+2}{1-r^2/R^2} - \frac{j(j+1)}{r^2(1-r^2/R^2)} \right] f_{\epsilon j} = 0 . \tag{24}$$

At the region far from the horizon $r \ll R$, the last equation reduces to

$$\left[\frac{d^2}{dR^2} + \frac{2}{R}\frac{d}{dR} + \epsilon^2 - M^2 - \frac{j(j+1)+2}{R^2} - \frac{j(j+1)}{R^2} \right] \bar{f}_{\epsilon j} = 0 . \tag{25}$$

So, we immediately conclude that eq. (23) coincides with (25) only for solutions with quantum numbers obeying the following restriction

$$\epsilon^2 - M^2 \gg \frac{j^2}{R^2} . \tag{26}$$

In usual units, this inequality reads

$$E = \mu\, mc^2, \qquad \lambda = \frac{\hbar}{mc}\,, \qquad \frac{\lambda^2}{R^2} \sim 10^{-80}\,, \qquad \mu^2 - 1 \gg \frac{\lambda^2}{R^2}\, j^2\,. \tag{27}$$

Instead, in massless case we have

$$\frac{R^2\omega^2}{c^2} \gg j^2 \qquad \text{or} \qquad \frac{R^2 4\pi^2}{\lambda^2} \gg j^2\,. \tag{28}$$

So we can state that the above relation (22) is a good approximation at a finite R only for quantum numbers obeying (27)–(28).

One additional point should be emphasized, the radial equation in de Sitter space can be transformed to the form of the Schrödinger like equation with an effective barrierless potential. Indeed, in the variable r^* eq. (24) reduces to

$$\left[\frac{d^2}{dr^{*2}} + \epsilon^2 - U(r^*)\right] G(r^*) = 0\,,$$
$$U(r^*) = \frac{1-r^2}{R^2}\left[4(1-r) \;+\; \frac{r}{1+r} \;+\; m^2R^2 \;+\; \frac{j(j+1)}{r^2}\right]\,. \tag{29}$$

It is easily verified that this potential corresponds to attractive force in all space

$$F_{r^*} \equiv -\frac{dU}{dr^*} = \frac{1-r^2}{R^2} \;+\; \left[2r\left(\frac{j(j+1)}{r^2} \;+\; m^2R^2 + \;\frac{r}{1+r}\right.\right.$$
$$\left.\left. +\, 4(1-r)\right) \;+\; (1-r^2)\left(\frac{2j(j+1)}{r^3} \;+\; 4 - \frac{1}{(1+r)^2}\right)\right] > 0\,.$$

At the horizon, $r^* \to \infty$, the potential function $U(r^*)$ tends to zero, so $G(r^*) \sim \exp(\pm i\epsilon r^*)$.

The form of the effective Shcrödinger equation modeling a particle in the de Sitter space indicates that the problem of calculation of the reflection coefficient in the system should not be even stated. However, in a number of publications such a problem has been treated and solved. So we should reconsider these calculations and results obtained. Significant steps of our approach are given below:

The existing in literature calculations of non-zero reflection coefficients $R_{\epsilon j}$ were based on the usage of the approximate formula for $U^{out}_{run}(R)$ for the region far from horizon (21). However, as noted above, the formula used is a good approximation only for solutions specified by (26), that is when $j \ll \epsilon R$.

It can be shown that the formula existing in the literature gives a trivial result when taking into account this restriction. The scheme of calculation (more detail see in [11]) is described below.

The first step consists in the use of the asymptotic formula for the Bessel functions: when $x \gg \nu^2$ we have

$$J(x) \sim \frac{\Gamma(2\nu+1)\; 2^{-2\nu-1/2}}{\Gamma(\nu+1)\;\Gamma(\nu+1/2)}\;\frac{1}{\sqrt{x}}\;\left[\;\exp\left(+i(x - \frac{\pi}{2}(\nu+\frac{1}{2}))\right) \;+\; \exp\left(-i(x - \frac{\pi}{2}(\nu+\frac{1}{2}))\right)\;\right],$$

so when $j < j^2 \ll \epsilon R << \epsilon R$ we derive

$$U^{out}_{run}(R) \sim \Big[\frac{e^{+i\epsilon R}}{\epsilon R} \Big(C_1 \ \exp(-i\frac{\pi}{2}(p+\frac{1}{2})) \ + \ C_2 \ \exp(-i\frac{\pi}{2}(-p+\frac{1}{2}) \Big) \\ + \frac{e^{-i\epsilon R}}{\epsilon R} \Big(C_1 \ \exp(+i\frac{\pi}{2}(p+\frac{1}{2})) \ + \ C_2 \ \exp(+i\frac{\pi}{2}(-p+\frac{1}{2})) \Big) \Big] \ , \tag{30}$$

where C_1 and C_2 are given by

$$C_1 = \frac{\Gamma(a+b+1-c)\ \Gamma(1-c)}{\Gamma(b-c+1)\ \Gamma(a-c+1)} \ \frac{2^{-j-1}\ \Gamma(2p+1)}{(\epsilon R)^j\ \Gamma(p+1/2)} \ , \\ C_1 = \frac{\Gamma(a+b+1-c)\ \Gamma(c-1)}{\Gamma(a)\ \Gamma(b)} \ \frac{(\epsilon R)^{j+1}\ \Gamma(-2p+1)}{2^{-j}\ \Gamma(p+1/2)} \ . \tag{31}$$

The reflection coefficient $R_{\epsilon j}$ is determined by the coefficients at $e^{-i\epsilon R}/\epsilon R$ and $e^{+i\epsilon R}/\epsilon R$. It is the matter of simple calculation to verify that when $\epsilon R \gg j$, the coefficient $R_{\epsilon j}$ is precisely zero

$$\epsilon R \gg j \ , \qquad R_{\epsilon j} \equiv 0 \ . \tag{32}$$

This conclusion is consistent with the analysis performed above.

III. SERIES EXPANSION ON A PARAMETER R^{-2} OF THE EXACT SOLUTIONS AND CALCULATION OF THE REFLECTION COEFFICIENT

In Section 3 we demonstrate that the basic instructive definition for the calculation of the reflection coefficient in de Sitter model, being grounded exclusively on the use of zero order approximation $\Phi^{(0)}(r)$ in the expansion of a particle wave function in a series of the form (1), cannot be extended on accounting for contributions of higher order terms.

Let us start with the solution, the wave running to the horizon

$$U^{out}(z) = \Gamma(a+b-c+1)\ [\ \alpha\ F(z) + \beta\ G(z)\ ] \ , \\ \alpha = \frac{\Gamma(1-c)}{\Gamma(b-c+1)\Gamma(a-c+1)} \ , \qquad \beta = \frac{\Gamma(c-1)}{\Gamma(a)\Gamma(b)} \ , \tag{33}$$

where

$$F(z) = z^{(p-1/2)/2}(1-z)^{-i\epsilon/2} F(a,b,c;z) \ , \qquad c = j+3/2 = 1+p \ , \\ a = \frac{1+p-i\epsilon+i\sqrt{m^2-1/4}}{2} \ , \qquad b = \frac{1+p-i\epsilon-i\sqrt{m^2-1/4}}{2} \ , \tag{34}$$

and

$$G(z) = z^{(-p-1/2)/2}(1-z)^{+i\epsilon/2} F(a-c+1, b-c+1, 2-c; z) \ , \qquad 2-c = 1-p \ , \\ a-c+1 = \frac{1-p-i\epsilon+i\sqrt{m^2-1/4}}{2} \ , \qquad b-c+1 = \frac{1-p-i\epsilon-i\sqrt{m^2-1/4}}{2} \ . \tag{35}$$

For the following we need the expressions for all quantities in usual units. It is convenient to change slightly the designation: now R stands for the curvature radius

$$z = \frac{r^2}{R^2}\,, \qquad \epsilon = \frac{ER}{\hbar c} = \mu\frac{R}{\lambda}\,, \qquad E = \mu\, Mc^2\,, \qquad m = \frac{McR}{\hbar} = \frac{R}{\lambda}\,, \qquad \lambda = \frac{\hbar}{Mc}\,. \tag{36}$$

Now, the relations (33)–(35) read

$$F(z) = R^{-p+1/2}\,\frac{r^p}{\sqrt{r}}\left(1-\frac{r^2}{R^2}\right)^{-i\mu R/2\lambda} F(a,b,c;\,\frac{r^2}{R^2})\,, \qquad c = 1+p\,,$$
$$a = \frac{1}{2}\left(1+p-i\mu\frac{R}{\lambda}+i\sqrt{\frac{R^2}{\lambda^2}-\frac{1}{4}}\right), \qquad b = \frac{1}{2}\left(1+p-i\mu\frac{R}{\lambda}-i\sqrt{\frac{R^2}{\lambda^2}-\frac{1}{4}}\right); \tag{37}$$

and

$$G(z) = R^{p+1/2}\,\frac{r^{-p}}{\sqrt{r}}\left(1-\frac{r^2}{R^2}\right)^{+i\mu R/2\lambda} F(a-c+1,b-c+1,2-c;\,\frac{r^2}{R^2})\,, \qquad 2-c = 1-p\,,$$
$$a-c+1 = \frac{1}{2}\left(1-p-i\mu\frac{R}{\lambda}+i\sqrt{\frac{R^2}{\lambda^2}-\frac{1}{4}}\right), \quad b-c+1 = \frac{1}{2}\left(1-p-i\mu\frac{R}{\lambda}-i\sqrt{\frac{R^2}{\lambda^2}-\frac{1}{4}}\right). \tag{38}$$

The task consists in obtaining the approximate expressions for $F(r)$ and $G(r)$ in the region far from horizon, $r \ll R$; first let us preserve the leading and next to leading terms (we have a natural small parameter λ/R).

First, let us consider the exponential factors

$$\left(1-\frac{r^2}{R^2}\right)^{\pm i\mu R/2\lambda} = \exp\left[\pm i\frac{\mu R}{2\lambda}\ln(1-\frac{r^2}{R^2})\right] = \cos\left[\frac{\mu R}{2\lambda}\ln(1-\frac{r^2}{R^2})\right] \pm i\sin\left[\frac{\mu R}{2\lambda}\ln(1-\frac{r^2}{R^2})\right]. \tag{39}$$

Using the expansion for the logarithmic function

$$\ln(1-x) = -(x+\frac{x^2}{2}+\frac{x^3}{3}+...)\,, \qquad \ln(1-\frac{r^2}{R^2}) = -\left(\frac{r^2}{R^2}+\frac{1}{2}\frac{r^4}{R^4}+\frac{1}{3}\frac{r^6}{R^6}+...\right)\,,$$

we get (assuming that $r^2 \ll \lambda R$)

$$\left(1-\frac{r^2}{R^2}\right)^{\pm i\mu R/2\lambda} = \cos\frac{\mu R}{2\lambda}\left(\frac{r^2}{R^2}+\frac{1}{2}\frac{r^4}{R^4}+\frac{1}{3}\frac{r^6}{R^6}+...\right)$$
$$\mp i\sin\frac{\mu R}{2\lambda}\left(\frac{r^2}{R^2}+\frac{1}{2}\frac{r^4}{R^4}+\frac{1}{3}\frac{r^6}{R^6}+...\right) \approx \left(1-\frac{\mu^2 r^4}{8\lambda^2 R^2}\right) \mp i\,\frac{\mu R}{2\lambda}\left(\frac{r^2}{R^2}+\frac{1}{2}\frac{r^4}{R^4}\right). \tag{40}$$

Terms in eq. (40) can be written in descending order

$$\left(1-\frac{r^2}{R^2}\right)^{\pm i\mu R/2\lambda} \approx 1 \mp i\mu\,\frac{r^2}{2\lambda R} - \frac{\mu^2}{2}\,\frac{r^4}{4\lambda^2 R^2} \mp i\mu\,X\,\frac{r^4}{4\lambda^2 R^2}\,,$$
$$X = \frac{\lambda}{R} \ll 1\,, \qquad \frac{r^2}{2\lambda R} \ll 1\,. \tag{41}$$

Now we turn to the hypergeometric function $F(a,b,c;z)$ from (37). Because $R \sim 10^{30}, \lambda \sim 10^{-12}$, one can use the approximation of a leading and two next order terms in the expressions for the following parameters

$$a = \frac{1}{2}\left(1+p-i\mu\frac{R}{\lambda}+i\frac{R}{\lambda}\sqrt{1-\frac{\lambda^2}{4R^2}}\right) = \frac{1+p}{2} - i\frac{\mu-1}{2}\frac{R}{\lambda} - i\frac{\lambda}{16R}\,,$$
$$b = \frac{1}{2}\left(1+p-i\mu\frac{R}{\lambda}-i\frac{R}{\lambda}\sqrt{1-\frac{\lambda^2}{4R^2}}\right) = \frac{1+p}{2} - i\frac{\mu+1}{2}\frac{R}{\lambda} + i\frac{\lambda}{16R}\,. \tag{42}$$

Then, the hypergeometric function is given as

$$F(a,b,c;\ \frac{r^2}{R^2}) = 1 + \frac{1}{R^2}\frac{ab}{c}r^2 + \frac{1}{2!}\frac{1}{R^4}\frac{a(a+1)b(b+1)}{c(c+1)}(r^2)^2$$
$$+\frac{1}{3!}\frac{1}{R^6}\frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}(r^2)^3 + ...$$
$$+\frac{1}{n!}\frac{1}{R^{2n}}\frac{a(a+1)(a+2)...(a+n-1)b(b+1)(b+2)..(b+n-1)}{c(c+1)(c+2)...(c+n-1)}(r^2)^n + ... \tag{43}$$

It is convenient to introduce a shortening notation for small quantity $X = \lambda/R$), then a typical term is represented as

$$\frac{1}{R^2}\frac{(a+n)(b+n)}{(c+n)} \approx \frac{1}{p+1+n}\times$$
$$\times\frac{-i(\mu-1)}{2\lambda}\left(1+i\frac{1+p+2n}{\mu-1}\,X+\frac{X^2}{8(\mu-1)}\right)\frac{-i(\mu+1)}{2\lambda}\left(1+i\frac{1+p+2n}{\mu+1}\,X-\frac{X^2}{8(\mu+1)}\right)\approx$$

$$\approx \frac{1}{p+1+n}\left(-\frac{\mu^2-1}{4\lambda^2}\right)\left[1+\frac{2i\mu}{\mu^2-1}(1+p+2n)\,X - \frac{(1+p+2n)^2-1/4}{\mu^2-1}\,X^2\right]\,. \tag{44}$$

Below we will use the notation

$$k^2 = \frac{\mu^2-1}{\lambda^2}\,,$$

then

$$\frac{1}{R^2}\frac{(a+n)(b+n)}{(c+n)} \approx \frac{1}{p+1+n}\left(-\frac{k^2}{4}\right)\left[1+\frac{2i\mu}{\mu^2-1}(1+p+2n)\,X - \frac{(1+p+2n)^2-1/4}{\mu^2-1}\,X^2\right].$$

Thus, we have the following approximate expressions for the first few terms in the series

$$\frac{r^2}{R^2}\frac{ab}{c} \approx \frac{1}{p+1}\left(-\frac{k^2r^2}{4}\right)\left[1+\frac{2i\mu}{\mu^2-1}(1+p)\;X-\frac{(1+p)^2-1/4}{\mu^2-1}\;X^2\right],$$

$$\frac{r^2}{R^2}\frac{(a+1)(b+1)}{(c+1)} \approx \frac{1}{p+2}\left(-\frac{k^2r^2}{4}\right)\left[1+\frac{2i\mu}{\mu^2-1}(1+p+2\times 1)\;X-\frac{(1+p+2\times 1)^2-1/4}{\mu^2-1}\;X^2\right],$$

$$\frac{r^2}{R^2}\frac{(a+2)(b+2)}{(c+2)} \approx \frac{1}{p+3}\left(-\frac{k^2r^2}{4}\right)\left[1+\frac{2i\mu}{\mu^2-1}(1+p+2\times 2)\;X-\frac{(1+p+2\times 2)^2-1/4}{\mu^2-1}\;X^2\right],$$

$$\frac{r^2}{R^2}\frac{(a+3)(b+3)}{(c+3)} \approx \frac{1}{p+4}\left(-\frac{k^2r^2}{4}\right)\left[1+\frac{2i\mu}{\mu^2-1}(1+p+2\times 3)\;X-\frac{(1+p+2\times 3)^2-1/4}{\mu^2-1}\;X^2\right],$$

$$\frac{r^2}{R^2}\frac{(a+4)(b+4)}{(c+4)} \approx \frac{1}{p+5}\left(-\frac{k^2r^2}{4}\right)\left[1+\frac{2i\mu}{\mu^2-1}(1+p+2\times 4)\;X-\frac{(1+p+2\times 4)^2-1/4}{\mu^2-1}\;X^2\right],$$

....

$$\frac{r^2}{R^2}\frac{(a+n)(b+n)}{(c+n)} \approx \frac{1}{p+1+n}\left(-\frac{k^2r^2}{4}\right)\left[1+\frac{2i\mu}{\mu^2-1}(1+p+2n)\;X-\frac{(1+p+2n)^2-1/4}{\mu^2-1}\;X^2\right].$$

Now we are ready to write down the expressions for the coefficients of the hypergeometric series preserving only leading and two next order terms

$$\frac{1}{1!}\frac{r^2}{R^2}\frac{ab}{c} \approx \frac{1}{p+1}\left(-\frac{k^2r^2}{4}\right)\left[1+\frac{2i\mu}{\mu^2-1}(1+p)\;X-\frac{(1+p)^2-1/4}{\mu^2-1}\;X^2\right],$$

$$\frac{1}{2!}\frac{(r^2)^2}{(R^2)^2}\frac{ab(a+1)(b+1)}{c(c+1)} \approx$$
$$\approx \left(-\frac{k^2r^2}{4}\right)^2\frac{1}{2!(p+1)(p+2)}\left\{1+\frac{2i\mu}{\mu^2-1}[(1+p)+(1+p+2\times 1)]\;X-\right.$$
$$\left.-X^2\left[\frac{4\mu^2}{(\mu^2-1)^2}(1+p)(1+p+2\times 1)+\frac{(1+p)^2-1/4}{\mu^2-1}+\frac{(1+p+2\times 1)^2-1/4}{\mu^2-1}\right]\right\},$$

$$\frac{1}{3!}\frac{(r^2)^3}{(R^2)^3}\frac{ab(a+1)(b+1)(a+2)(b+2)}{c(c+1)(c+2)} \approx \left(-\frac{k^2r^2}{4}\right)^3\frac{1}{3!(p+1)(p+2)(p+3)}\times$$
$$\times\left\{1+X\;\frac{2i\mu}{\mu^2-1}[(1+p)+(1+p+2\times 1)+(1+p+2\times 2)]-\right.$$
$$-X^2\left[\frac{4\mu^2}{(\mu^2-1)^2}[(1+p)+(1+p+2\times 1)](1+p+2\times 2)+\right.$$
$$\left.\left.+\frac{(1+p)^2-1/4}{\mu^2-1}+\frac{(1+p+2\times 1)^2-1/4}{\mu^2-1}+\frac{(1+p+2\times 2)^2-1/4}{\mu^2-1}\right]\right\},$$

$$\frac{1}{4!}\frac{(r^2)^4}{(R^2)^4}\frac{ab(a+1)(b+1)(a+2)(b+2)(a+3)(b+3)}{c(c+1)(c+2)(c+3)} \approx$$

$$\approx \left(-\frac{k^2r^2}{4}\right)^4 \frac{1}{4!(p+1)(p+2)(p+3)(p+4)} \times$$

$$\times \Big\{ 1 + X\,\frac{2i\mu}{\mu^2-1}[(1+p)+(1+p+2\times 1)+(1+p+2\times 2)+(1+p+2\times 3)]-$$

$$-X^2\left[\frac{4\mu^2}{(\mu^2-1)^2}[(1+p)+(1+p+2\times 1)+(1+p+2\times 2)](1+p+2\times 3)+\right.$$

$$\left.+\frac{(1+p)^2-1/4}{\mu^2-1}+\frac{(1+p+2\times 1)^2-1/4}{\mu^2-1}+\frac{(1+p+2\times 2)^2-1/4}{\mu^2-1}+\frac{(1+p+2\times 3)^2-1/4}{\mu^2-1}\right]\Big\} \ ,$$

....

$$\frac{1}{n!}\frac{(r^2)^n}{(R^2)^n}\frac{ab\ (a+1)(b+1)....(a+n-1)(b+n-1)}{c(c+1)...(c+n)} \approx \left(-\frac{k^2r^2}{4}\right)^n \frac{1}{n!(p+1)(p+2)...(p+n)}\times$$

$$\times \Big\{ 1 + X\ \frac{2i\mu}{\mu^2-1}\ [(1+p)+(1+p+2\times 1)+(1+p+2\times 2)+...+(1+p+2\times(n-1))]-$$

$$-X^2\left[\frac{4\mu^2}{(\mu^2-1)^2}[(1+p)+(1+p+2\times 1)+...+(1+p+2\times(n-2))]\ (1+p+2\times(n-1))+\right.$$

$$\left.+\frac{(1+p)^2-1/4}{\mu^2-1}+\frac{(1+p+2\times 1)^2-1/4}{\mu^2-1}+...+\frac{(1+p+2\times(n-1))^2-1/4}{\mu^2-1}\right]\Big\} \ .$$

Thus, initial exact hypergeometric function can be approximated by the sum of three series

$$\bar{F} = F(a,b,c;\ \frac{r^2}{R^2}) = \bar{F}_0(r) + X\ \bar{F}_1(r) + X^2\ \bar{F}_2(r)\ . \tag{45}$$

The leading series $\bar{F}_0(r)$, in fact, reduces to the Bessel function

$$\bar{F}_0(r) = 1 + \frac{(ikr/2)^2}{n!(p+1)} + \frac{(ikr/2)^4}{2!(p+1)(p+2)} + ... + \frac{(ikr/2)^{2n}}{n!(p+1)(p+2)...(p+n)} + ...$$
$$= \Gamma(p+1)\sum_{n=0}^{\infty}\frac{(ikr/2)^{2n}}{n!\Gamma(p+1+n)} = \Gamma(1+p)\left(\frac{kr}{2}\right)^{-p} J_p(kr)\ , \tag{46}$$

where

$$J_p(x) = (\frac{x}{2})^p \sum_0^{\infty}\frac{(-x^2/4)^n}{n!\ \Gamma(p+1+n)}\ , \qquad x = kr\ .$$

The second series is given by

$$X\ \bar{F}_1(r) = X\ \frac{2i\mu}{\mu^2-1}\Big\{(-k^2r^2/4)\frac{p+1}{p+1} + (-k^2r^2/4)^2\frac{[(1+p)+(1+p+2\times 1)]}{2!(p+1)(p+2)} +$$

$$+(-k^2r^2/4)^3 \frac{[(1+p)+(1+p+2\times 1)+(1+p+2\times 2)]}{3!(p+1)(p+2)(p+3)} +$$
$$+(-k^2r^2/4)^4 \frac{[(1+p)+(1+p+2\times 1)+(1+p+2\times 2)+(1+p+2\times 3)]}{4!(p+1)(p+2)(p+3)(p+4)} + ...$$
$$(-k^2r^2/4)^n \frac{[(1+p)+(1+p+2\times 1)+...+(1+p+2\times (n-1))]}{n!(p+1)(p+2)...(p+n)} \Big\} \; ; \qquad (47)$$

that is

$$X\bar{F}_1(r) = X\frac{2i\mu}{\mu^2-1}\left(\frac{-k^2r^2}{4}\right)\left\{1+\left(\frac{-k^2r^2}{4}\right)\frac{[(1+p)+(1+p+2\times 1)]}{2!(p+1)(p+2)}+\right.$$
$$+\left(\frac{-k^2r^2}{4}\right)^2 \frac{[(1+p)+(1+p+2\times 1)+(1+p+2\times 2)]}{3!(p+1)(p+2)(p+3)} +$$
$$+\left(\frac{-k^2r^2}{4}\right)^3 \frac{[(1+p)+(1+p+2\times 1)+(1+p+2\times 2)+(1+p+2\times 3)]}{4!(p+1)(p+2)(p+3)(p+4)} + ...$$
$$\left. ...+\left(\frac{-k^2r^2}{4}\right)^n \frac{[(1+p)+(1+p+2\times 1)+...+(1+p+2\times n)]}{(n+1)!(p+1)(p+2)...(p+n+1)}\right\} \; ; \qquad (48)$$

or shorter

$$X\bar{F}_1(r) = X\frac{2i\mu}{\mu^2-1}\,\Gamma(p+1)\left(\frac{-k^2r^2}{4}\right)\;\sum_{n=0}^{\infty}\left(\frac{-k^2r^2}{4}\right)^n\;\frac{[(1+p)+...+(1+p+2\times n)]}{(n+1)!\;\Gamma(p+2+n)}\;. \qquad (49)$$

Using known sums

$$[(1+p)+...+(1+p+2\times n) = (1+p)(n+1)+2(1+2+3+...n) =$$

$$= (p+1)(1+n)+2\frac{(1+n)n}{2} = (n+1)(n+1+p)$$

we derive

$$X\bar{F}_1(r) = X\frac{2i\mu}{\mu^2-1}\,\Gamma(p+1)\left(\frac{-k^2r^2}{4}\right)\;\sum_{n=0}^{\infty}(-k^2r^2/4)^n\;\frac{(n+1)(n+1+p)}{(n+1)!\;\Gamma(p+2+n)} =$$
$$= X\frac{2i\mu}{\mu^2-1}\,\Gamma(p+1)\left(\frac{-k^2r^2}{4}\right)\;\sum_{n=0}^{\infty}\;\frac{(-k^2r^2/4)^n}{n!\;\Gamma(p+1+n)} =$$
$$= X\left(\frac{-k^2r^2}{4}\right)\frac{2i\mu}{\mu^2-1}\,\Gamma(p+1)\left(\frac{kr}{2}\right)^{-p}\;J_p(kr) = X\left(\frac{-k^2r^2}{4}\right)\frac{2i\mu}{\mu^2-1}\;\bar{F}_0(r) \qquad (50)$$

Thus, the approximation (45) can be presented as follows

$$\bar{F} = \bar{F}_0(r)+X\bar{F}_1(r)+X^2\bar{F}_2(r) = \bar{F}_0(r)+X\;\frac{2i\mu}{\mu^2-1}\;(\frac{-k^2r^2}{4})\;\bar{F}_0(r)+X^2\bar{F}_2(r)\;, \qquad (51)$$

Similar relations can be derived for the hypergeometric series $\bar{G}(r)$:

$$\bar{G} = F(a-c+1, b-c+1, 2-c;\;\frac{r^2}{R^2}) = \bar{G}_0(r)+X\;\bar{G}_1(r)+X^2\;\bar{G}_2(r)\;. \qquad (52)$$

The leading term again reduces to the Bessel function

$$\bar{G}_0(r) = \Gamma(1-p)\left(\frac{kr}{2}\right)^{+p} J_{-p}(kr) \ . \tag{53}$$

Next order term is given by

$$X\bar{G}_1(r) = X\left(\frac{-k^2r^2}{4}\right)\frac{2i\mu}{\mu^2-1}\bar{G}_0(r) \tag{54}$$

So, the approximation (52) is presented as

$$\bar{G} = \bar{G}_0(r) + X\bar{G}_1(r) + X^2\bar{G}_2(r) = \bar{G}_0(r) + X\ \frac{2i\mu}{\mu^2-1}\ (\frac{-k^2r^2}{4})\ \bar{G}_0(r) + X^2\bar{G}_2(r)\ , \tag{55}$$

Now let us consider the whole function

$$F(z) = R^{-p+1/2}\ \frac{r^p}{\sqrt{r}}\left(1-\frac{r^2}{R^2}\right)^{-i\mu R/2\lambda} F(a,b,c;\ \frac{r^2}{R^2}) =$$

$$= R^{-p+1/2}\ \frac{r^p}{\sqrt{r}}\left[1 + i\mu\ \frac{r^2}{2\lambda R} - \frac{\mu^2}{2}\ \frac{r^4}{4\lambda^2R^2} + i\mu\ X\ \frac{r^4}{4\lambda^2R^2}\right]\times$$

$$\times\left[\bar{F}_0(r) + \frac{2i\mu}{\mu^2-1}\ (\frac{-k^2r^2}{4})\ X\bar{F}_0(r) + X^2\bar{F}_2(r)\right] =$$

$$= R^{-p+1/2}\ \frac{r^p}{\sqrt{r}}\Bigg[\bar{F}_0(r) + \frac{2i\mu}{\mu^2-1}\ (\frac{-k^2r^2}{4})\ X\bar{F}_0(r) + X^2\bar{F}_2(r) +$$

$$+ i\mu\ \frac{r^2}{2\lambda R}\bar{F}_0(r) + i\mu\ \frac{r^2}{2\lambda R}\frac{2i\mu}{\mu^2-1}\ (\frac{-k^2r^2}{4})\ X\bar{F}_0(r) + i\mu\ \frac{r^2}{2\lambda R}X^2\bar{F}_2(r) -$$

$$- \frac{\mu^2}{2}\ \frac{r^4}{4\lambda^2R^2}\bar{F}_0(r) - \frac{\mu^2}{2}\ \frac{r^4}{4\lambda^2R^2}\frac{2i\mu}{\mu^2-1}\ (\frac{-k^2r^2}{4})\ X\bar{F}_0(r) - \frac{\mu^2}{2}\ \frac{r^4}{4\lambda^2R^2}X^2\bar{F}_2(r) +$$

$$+ i\mu\ X\ \frac{r^4}{4\lambda^2R^2}\bar{F}_0(r) + i\mu\ X\ \frac{r^4}{4\lambda^2R^2}\frac{2i\mu}{\mu^2-1}\ (\frac{-k^2r^2}{4})\ X\bar{F}_0(r) + i\mu\ X\ \frac{r^4}{4\lambda^2R^2}X^2\bar{F}_2(r)\Bigg]$$

$$= R^{-p+1/2}\ \frac{r^p}{\sqrt{r}}\Bigg[\bar{F}_0(r) - i\mu\ \frac{r^2}{2\lambda R}\ \bar{F}_0(r) + X^2\bar{F}_2(r) +$$

$$+ i\mu\frac{r^2}{2\lambda R}\bar{F}_0(r) + \mu^2(\frac{r^2}{2\lambda R})^2\bar{F}_0(r) + i\mu\ \frac{r^2}{2\lambda R}\ X^2\bar{F}_2(r) -$$

$$-\frac{\mu^2}{2}\,(\frac{r^2}{2\lambda R})^2\;\bar{F}_0(r)+\frac{i\mu^3}{2}(\frac{r^2}{2\lambda R})^3\;\bar{F}_0(r)-\frac{\mu^2}{2}\,(\frac{r^2}{2\lambda R})^2\;X^2\bar{F}_2(r)+$$

$$\left.+i\mu\;X\;(\frac{r^2}{2\lambda R})^2\;\bar{F}_0(r)+\mu^2(\frac{r^2}{2\lambda R})^3\bar{F}_0(r)+i\mu\;(\frac{r^2}{2\lambda R})^2X^3\bar{F}_2(r)\right]$$

Preserving only first two terms we have

$$F(z)=R^{-p+1/2}\;\frac{r^{+p}}{\sqrt{r}}\left[\bar{F}_0(r)+\frac{1}{8}\mu^2\frac{r^2r^2}{\lambda^2R^2}\;\bar{F}_0(r)+\frac{\lambda^2}{R^2}\bar{F}_2(r)\right]. \tag{56}$$

Similarly, for $G(r)$ we obtain

$$G(z)=R^{+p+1/2}\;\frac{r^{-p}}{\sqrt{r}}\left[\bar{G}_0(r)+\frac{1}{8}\mu^2\frac{r^2r^2}{\lambda^2R^2}\;\bar{G}_0(r)+\frac{\lambda^2}{R^2}\bar{G}_2(r)\right]. \tag{57}$$

One should emphasize one feature of the expansions (56) and (57): these approximations are real valued as we must expect remembering on relations (10) and (13).

In the known method of determining and calculating the reflection coefficients in de Sitter model [3–6], authors used the only leading terms in approximations (56) and (57), because only these terms allows to separate elementary solutions of the form $e^{\pm ikr}$ – see (30).

Let us perform some additional calculations to clarify the problem. The functions $F(z)$ and $F(z)$ enter the expression for (to horizon) running wave

$$U^{out}(z)=\Gamma(a+b-c+1)[\;\alpha\;F(z)+\beta\;G(z)\;]\;,$$
$$\alpha=\frac{\Gamma(1-c)}{\Gamma(b-c+1)\Gamma(a-c+1)}\;,\qquad \beta=\frac{\Gamma(c-1)}{\Gamma(a)\Gamma(b)}\;, \tag{58}$$

where (introducing a very large parameter $Y=R/2\lambda$)

$$\alpha\approx\frac{\Gamma(-p)}{\Gamma[-i(\mu-1)Y+(1-p)/2)]\;\;\Gamma[-i(\mu+1)Y+(1-p)/2)]}\;,$$

$$\beta\approx\frac{\Gamma(+p)}{\Gamma[-i(\mu-1)Y+(1+p)/2)]\;\;\Gamma[-i(\mu+1)Y+(1+p)/2)]}\;. \tag{59}$$

For all physically reasonable values of quantum numbers j (not very high ones) and values of μ (different from the critical value $\mu=1$ and not too high ones – they correspond in fact to energies of a particle in units of the rest energy) the argument of Γ-functions in (59) are complex-valued with very large imaginary parts.

Let us multiply the given solution (58) by a special factor A which permits us to distinguish small and large parts in this expansion:

$$A=\frac{\Gamma(a-p/2+1/4)\Gamma(b-p/2+1/4)}{\Gamma(a+b-c+1)}\;. \tag{60}$$

So, instead of (58), we get

$$AU^{out}(z) = \alpha' \, F(z) + \beta' \, G(z) \; ,$$
$$\alpha' = \Gamma(1-c) \frac{\Gamma(a-p/2+1/4)\Gamma(b-p/2+1/4)}{\Gamma(b-c+1)\Gamma(a-c+1)} \; ,$$
$$\beta' = \Gamma(c-1) \frac{\Gamma(a-p/2+1/4)\Gamma(b-p/+1/4)}{\Gamma(a)\Gamma(b)} \; . \tag{61}$$

Instead of (59), the expressions for α' and β' are

$$\alpha' \approx \Gamma(-p) \; \frac{\Gamma[-i(\mu-1)Y+1/4]}{\Gamma[-i(\mu-1)Y+(1-p)/2)]} \; \frac{\Gamma[-i(\mu+1)Y+1/4]}{\Gamma[-i(\mu+1)Y+(1-p)/2)]} \; ,$$
$$\beta' \approx \Gamma(+p) \; \frac{\Gamma[-i(\mu-1)Y+1/4]}{\Gamma[-i(\mu-1)Y+(1+p)/2)]} \; \frac{\Gamma[-i(\mu+1)Y+1/4]}{\Gamma[-i(\mu+1)Y+(1+p)/2)]} \; . \tag{62}$$

Allowing for identities

$$\Gamma(-p) = -\frac{\pi}{\sin p\pi} \frac{1}{\Gamma(1+p)} \; , \qquad \Gamma(p) = +\frac{\pi}{\sin p\pi} \frac{1}{\Gamma(1-p)} \; , \tag{63}$$

α' and β' (62) are transformed into

$$\alpha' \approx -\frac{\pi}{\sin p\pi} \frac{1}{\Gamma(1+p)} \; \frac{\Gamma[-i(\mu-1)Y+1/4]}{\Gamma[-i(\mu-1)Y+(1-p)/2)]} \; \frac{\Gamma[-i(\mu+1)Y+1/4]}{\Gamma[-i(\mu+1)Y+(1-p)/2)]} \; ,$$
$$\beta' \approx +\frac{\pi}{\sin p\pi} \frac{1}{\Gamma(1-p)} \; \frac{\Gamma[-i(\mu-1)Y+1/4]}{\Gamma[-i(\mu-1)Y+(1+p)/2)]} \; \frac{\Gamma[-i(\mu+1)Y+1/4]}{\Gamma[-i(\mu+1)Y+(1+p)/2)]} \; . \tag{64}$$

Now we have to take into account the asymptotic formula for Γ-function

$$\frac{\Gamma(z+A)}{\Gamma(z+B)} = z^{A-B} \left(1 + \frac{1}{z} \; \frac{(A-B)(A+B+1)}{2} + ... \right) \; , \qquad \mid \arg z \mid < \pi \; , \qquad \mid z \mid \to \infty \; ; \tag{65}$$

then (remembering that $Y = R/2\lambda$)

$$\frac{\Gamma[-i(\mu-1)Y+1/4]}{\Gamma[-i(\mu-1)Y+(1-p)/2)]} \approx \left[-i \frac{(\mu-1)}{2\lambda} R \right]^{p/2-1/4} \; \left[1 + \frac{2\lambda}{-i(\mu-1)R} \frac{(2p-1)(7-2p)}{32} \right] \; ,$$

$$\frac{\Gamma[-i(\mu+1)Y+1/4]}{\Gamma[-i(\mu+1)Y+(1-p)/2)]} \approx \left[-i \frac{(\mu+1)}{2\lambda} R \right]^{p/2-1/4} \left[1 + \frac{2\lambda}{-i(\mu+1)R} \frac{(2p-1)(7-2p)}{32} \right], \tag{66}$$

and

$$\frac{\Gamma[-i(\mu-1)Y+1/4]}{\Gamma[-i(\mu-1)Y+(1+p)/2)]} \approx \left[-i \frac{(\mu-1)}{2\lambda} R \right]^{-p/2-1/4} \; \left[1 + \frac{2\lambda}{-i(\mu-1)R} \frac{(-2p-1)(7+2p)}{32} \right] \; ,$$

$$\frac{\Gamma[-i(\mu+1)Y+1/4]}{\Gamma[-i(\mu+1)Y+(1+p)/2)]} \approx \left[-i \frac{(\mu+1)}{2\lambda} R \right]^{-p/2-1/4} \; \left[1 + \frac{2\lambda}{-i(\mu+1)R} \frac{(-2p-1)(7+2p)}{32} \right] \; . \tag{67}$$

Substituting (66) and (67) into (64), we obtain

$$\alpha' \approx -\left(\frac{\mu^2-1}{4\lambda^2}R^2\right)^{p/2-1/4} \quad (-1)^{p/2-1/4}\ \frac{\pi}{\sin p\pi}\frac{1}{\Gamma(1+p)} \ \times$$
$$\times \left[1+\frac{2\lambda}{-i(\mu-1)R}\frac{(2p-1)(7-2p)}{32}\right]\left[1+\frac{2\lambda}{-i(\mu+1)R}\frac{(2p-1)(7-2p)}{32}\right],$$

$$\beta' \approx +\left(\frac{\mu^2-1}{4\lambda^2}R^2\right)^{-p/2-1/4} \quad (-1)^{-p/2-1/4}\ \frac{\pi}{\sin p\pi}\frac{1}{\Gamma(1-p)} \ \times$$
$$\times \left[1+\frac{2\lambda}{-i(\mu-1)R}\frac{(-2p-1)(7+2p)}{32}\right]\left[1+\frac{2\lambda}{-i(\mu+1)R}\frac{(-2p-1)(7+2p)}{32}\right]. \qquad (68)$$

Preserving the only terms of first two orders we have

$$\alpha' \approx -\frac{\pi}{\sin p\pi}\frac{1}{\Gamma(1+p)}\left(\frac{k}{2}\right)^{+p-1/2} R^{+p-1/2}(-1)^{p/2-1/4}\left(1+i\frac{(2p-1)(7-2p)}{8}\frac{\mu}{\mu^2-1}\frac{\lambda}{R}\right),$$

$$\beta' \approx +\frac{\pi}{\sin p\pi}\frac{1}{\Gamma(1-p)}\left(\frac{k}{2}\right)^{-p-1/2} R^{-p-1/2}(-1)^{-p/2-1/4}\left(1-i\frac{(-2p-1)(7+2p)}{8}\frac{\mu}{\mu^2-1}\frac{\lambda}{R}\right). \qquad (69)$$

Substituting expressions (69) into the following expansion

$$AU^{out}(z) = \alpha'\ F(z) + \beta'\ G(z)\ ,$$
$$F(r) = R^{-p+1/2}\ \Gamma(1+p)\left(\frac{k}{2}\right)^{-p}\frac{1}{\sqrt{r}}J_p(kr)\ ,$$
$$G(r) \approx R^{p+1/2}\ \Gamma(1-p)\left(\frac{k}{2}\right)^{p}\frac{1}{\sqrt{r}}J_{-p}(kr)\ , \qquad (70)$$

we arrive at the following zero-order approximation

$$AU^{out}(z) = \psi_0^{out}(r) =$$

$$= -\frac{\pi}{\sin p\pi}\frac{1}{\Gamma(1+p)}\left(\frac{k}{2}\right)^{+p-1/2} R^{+p-1/2}(-1)^{p/2-1/4}R^{-p+1/2}\Gamma(1+p)\left(\frac{k}{2}\right)^{-p}\frac{1}{\sqrt{r}}J_p(kr)+$$

$$+\frac{\pi}{\sin p\pi}\frac{1}{\Gamma(1-p)}\left(\frac{k}{2}\right)^{-p-1/2} R^{-p-1/2}(-1)^{-p/2-1/4}R^{p+1/2}\Gamma(1-p)\left(\frac{k}{2}\right)^{p}\frac{1}{\sqrt{r}}J_{-p}(kr)\ ,$$

that is

$$AU^{out}(z) = \psi_0^{out}(r) =$$

$$= \frac{\pi}{\sin p\pi}\ \sqrt{\frac{2k}{r}}\left[\ -(-1)^{p/2-1/4}\ J_p(kr) +\ (-1)^{-p/2-1/4}\ J_{-p}(kr)\ \right] =$$
$$= -(-1)^{-p/2-1/4}\sqrt{\frac{2k}{r}}\ \ \frac{\pi}{\sin p\pi}\left[\ (-1)^p\ J_p(kr) - J_{-p}(kr)\ \right]\ ; \qquad (71)$$

which coincides with a spherical wave propagating in Minkowski space from the origin, expressed through the Hankel functions of the first kind

$$H_p^{(1)} = \frac{ip}{\sin p\pi} \left[(-1)^p J_p(kr) - J_{-p}(kr) \right] \ . \tag{72}$$

IV. CONCLUSION

The last but not the least mathematical remark should be given. All known quantum mechanical problems with potentials containing one barrier reduce to a second order differential equation with four singular points, the equation of Heun class. In particular, the most popular cosmological problem of that type is a particle in the Schwarzschild space-time background and it reduces to the Heun differential equation. Quantum mechanical problems of tunneling type are never linked to differential equation of hypergeometric type, equation with three singular points; but in the case of de Sitter model the wave equations for different fields, of spin 0, 1/2, and 1, after separation of variables are reduced to the second order differential equation with three singular points, and there exists no ground to search in these systems problems of tunneling class.

Acknowledgements

Authors are grateful to participants of seminar of Laboratory of theoretical physics, Institute of Physics of National Academy of Sciences of Belarus for stimulating discussion.

We wish to thank the Organizers of the XLVIII All-Russia conference on problems in Particle Physics, Plasma Physics, Condensed Matter, and Optoelectronics Russia, Moscow, 15-18 May 2012, dedicated to the 100-th anniversary of Professor Ya.P. Terletsky, for opportunity to give a talk on the subject.

[1] Hawking, S.W.: Particle creation by black holes. Commun. Math. Phys. **43**, 199–220 (1975)

[2] Hawking, S.W., Gibbons, G.W.: Cosmological event horizons, thermodynamics, and particle creation. Phys. Rev. D. **15**, 2738 – 2751 (1977)

[3] Lohiya, D., Panchapakesan, N.: Massless scalar field in a de Sitter universe and its thermal flux. J. Phys. A. **11**, 1963–1968 (1978)

[4] Lohiya, D., Panchapakesan, N.: Particle emission in the de Sitter universe for massless fields with spin. J. Phys. A. **12**, 533–539 (1979)

[5] Khanal, U., Panchapakesan, N.: Perturbation of the de Sitter-Schwarzchild universe with massless fields. Phys. Rev. D. **24**, 829–834 (1981)

[6] Khanal, U., Panchapakesan, N.: Production of massless particles in the de Sitter-Schwarzschild universe. Phys. Rev. D. **24**, 835–838 (1981)

[7] Otchik, V.S.: On the Hawking radiation of spin 1/2 particles in the de Sitter space-time. Class. Quantum Crav. **2**, 539–543 (1985)

[8] Bogush, A.A., Otchik, V.S., Red'kov, V.M.: Vector field in de Sitter space. Vesti AN NSSR. **1**, 58–62 (1986)

[9] Takashi Mishima, Akihiro Nakayama: Particle production in de Sitter spacetime. Progr. Theor. Phys. **77**, 218–222 (1987)

[10] Suzuki, H., Takasugi, E.: Absorption Probability of De Sitter Horizon for Massless Fields with Spin. Mod. Phys. Lett. A. **11**, 431–436 (1996)

[11] V.M. Red'kov, E.M. Ovsiyuk. On exact solutions for quantum particles with spin $S = 0, 1/2, 1$ and de Sitter event horizon. Ricerche di matematica. - 2011. - Vol. 60. no 1, P. 57-88.